Sharp pointwise estimates for Riesz potentials with bounded density

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Riesz, Björn och födelsedagen...

Björn, Grattis på födelsedagen!

Tack Vladimir!

Jag tänkte svara på det föregående mejlet, "I am retired since ten years (today!)", men jag avstod från det inom parentesen.

Vi pratade lite om Riesz förra veckan på SU. Någon frågade vem som var Frostmans handledare, och jag svarade M. Riesz. Och vi diskuterade saker om Åke Pleijel, pappa till Agneta Pleijel (författare, med bl.a. den intressanta boken "Sniglar och snö"). Om jag minns rätt så efterträddes M. Riesz av Åke Pleijel i Lund, och Åke gifte om sig med en av Riesz döttrar. Osv osv.

Bästa hälsningar,

Björn

2024/10/30

V.T. comment: "Berget på månens baksida", Swedish drama film about the life of the Russian mathematician Sofia Kovalevskaya, written by Agneta Pleijel

In calculus (including) PDEs, there are two principal ingredients: the Cauchy-Bunyakovsky-Schwarz inequality and integration by parts, with infinitely many variations.

(Mathematical folklore)

Some relevant contexts

Several natural contexts including the following:

- Moment and L-problem of moments (A.A. Markov, M. Krein, N. Akhiezer, M. Putinar)
- Exponential transform of quadrature domains and domain identification (M. Putinar, B. Gustafsson, G. Golub, P. Milanfar, V.T.)
- Exponential transform and regularity of free boundaries in two dimensions by M. Putinar, B. Gustafsson, *Ann. SNS Pisa*, 1998
- Riesz potentials and regularity theory (by D. Adams, L. Hedberg, G. Mingione)
- Moving-centre monotonicity formulas (Jonathan Zhu, J. Funct. Anal., 2018)
- Recently in "Moving monotonicity formulae for minimal submanifolds in constant curvature", by K. Naff, J.J. Zhu, link
- Moment indeterminateness: the Marcel Riesz variational principle, by David P. Kimsey, Mihai Putinar, arXiv:2307.16018 (2023) is a nice source for history, motivations and ideas

Two classical inequalities

The **Cauchy-Bunyakovsky-Schwarz inequality**: under some natural assumptions,

$$\left(\int_{\mathbb{R}} f(x)g(x)\,dx\right)^2 \leq \int_{\mathbb{R}} f(x)^2\,dx \cdot \int_{\mathbb{R}} g(x)^2\,dx$$

Proof follows from

$$0 \leq \int_{\mathbb{R}} (f(x) - t \cdot g(x))^2\,dx = \int_{\mathbb{R}} f^2(x)\,dx - 2t \int_{\mathbb{R}} f(x)g(x)\,dx + t^2 \int_{\mathbb{R}} g^2(x)\,dx$$

and the fact that the discriminant is non-positive.

The **Markov inequality**: for any positive continuous random variable ξ

$$a \cdot \Pr(\xi > a) \leq \mathrm{E}(\xi)$$

Proof.

$$a \cdot \int_{[a,\infty)} f_\xi(x)\,dx \leq \int_{\mathbb{R}} x f_\xi(x)\,dx$$

A couple of weird sharp inequalities

Theorem 1

For any measurable function $0 \le \rho(x) \le 1$, $x \in \mathbb{R}$, *with* **compact** *support* $\not\ni 0$ *there holds*

$$\sinh^2\left(\frac{1}{2}\int_{\mathbb{R}} \frac{\rho(x)}{|x|}dx\right) \le \frac{1}{4}\int_{\mathbb{R}} \rho(x)dx \int_{\mathbb{R}} \frac{\rho(x)}{x^2}dx, \tag{1}$$

$$\frac{1}{2}\left(\int_{\mathbb{R}} \rho(x)dx\right)^2 \le \tanh\left(\frac{1}{2}\int_{\mathbb{R}} \frac{\rho(x)}{|x|}dx\right) \cdot \int_{\mathbb{R}} |x|\rho(x)dx. \tag{2}$$

The inequalities are **sharp** *and attained iff* $\rho(x)$ *is a characteristic function of an interval* $[a, b]$ *with* $ab > 0$.

Remark. For $\rho = \chi_{[a,b]}$ the above inequalities become equalities:

$$\sinh^2\left(\frac{1}{2}\ln\frac{b}{a}\right) = \left(\frac{\frac{\sqrt{b}}{\sqrt{a}} - \frac{\sqrt{a}}{\sqrt{b}}}{2}\right)^2 = \frac{1}{4}(b-a)\cdot\frac{b-a}{ab},$$

$$\frac{1}{2}(b-a)^2 = \tanh\left(\frac{1}{2}\ln\frac{b}{a}\right)\cdot\frac{b^2-a^2}{2} = \frac{\frac{b}{a}-1}{\frac{b}{a}+1}\cdot\frac{b^2-a^2}{2}.$$

Why the above inequalities are strange? They are '**transcendental**'.

Indeed, looking ahead, I can tell you that if one consider the moments of positive degrees, then the corresponding (Markov type) inequalities are **algebraic** (in fact, polynomial). For example

$$\left(\int_{\mathbb{R}+} \rho(x)\,dx\right)^4 \leq 12\left[\left(\int_{\mathbb{R}+} \rho(x)\,dx\right)\left(\int_{\mathbb{R}+} x^2\rho(x)\,dx\right) - \left(\int_{\mathbb{R}} x\rho(x)\,dx\right)^2\right],$$

The inequality is sharp and again, attains for the characteristic function of an interval:

$$\begin{aligned}(b-a)^4 &\leq 12\left[(b-a)\cdot\frac{b^3-a^3}{3} - \frac{(b^2-a^2)^2}{4}\right] \\ &= 12\cdot\frac{(b-a)^2}{3\cdot 4}\left[4(a^2+ab+b^2) - 3(a^2+2ab+b^2)\right] \\ &= {\color{blue}(b-a)^4}\end{aligned}$$

Riesz potentials for bounded measures

Let $\rho : \mathbb{R}^n \to [0,1]$ be an meausarable (nonnegative bounded) function with compact support. Let us consider the Riesz potential of index α

$$(\mathcal{I}_\alpha \rho)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{n-\alpha}} d_\omega \mathbf{y} = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{\rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{n-\alpha}}\, d\mathbf{y}, \quad x \notin \operatorname{supp}(\rho)$$

where the integrals are normalized by

$$d_\omega \mathbf{y} := \frac{1}{\omega_n} d\mathbf{y} = \frac{1}{\omega_n} dy_1 \cdot \ldots \cdot dy_n,$$

ω_n being the n-dimensional Lebesgue measure of the **unit ball** in $\mathbb{R}^n$. Then

$$-\frac{1}{n-\alpha} (\nabla \mathcal{I}_\alpha \rho)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{(\mathbf{y}-\mathbf{x})\rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{n+2-\alpha}}\, d_\omega \mathbf{y}.$$

If $\mathbf{e_x} \in \mathbb{R}^n$ is the normalized $(-\nabla \mathcal{I}_\alpha \rho)(\mathbf{x})$ then

$$\frac{1}{n-\alpha} |(\nabla \mathcal{I}_\alpha \rho)(\mathbf{x})| = \int_{\mathbb{R}^n} \frac{\langle \mathbf{y}-\mathbf{x}; \mathbf{e_x}\rangle \cdot \rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{n+2-\alpha}}\, d_\omega \mathbf{y} = \int_{\mathbb{R}^n} \frac{\langle \mathbf{z}; \mathbf{e_x}\rangle \cdot \rho(\mathbf{z}+\mathbf{x})}{|\mathbf{z}|^{n+2-\alpha}}\, d_\omega \mathbf{z}$$

$$= \int_{\mathbb{R}^n} \frac{z_1}{|\mathbf{z}|^{n-(\alpha-2)}} \cdot \tilde{\rho}(\mathbf{z})\, d_\omega \mathbf{z}, \qquad \operatorname{supp} \tilde{\rho} \subset \mathbb{R}^n \setminus \{0\}$$

So, I will be interested in certain sharp point-wise estimates involving

$$\frac{1}{n-\alpha}\left|(\nabla \mathcal{I}_\alpha \rho)(\mathbf{x})\right| = \int_{\mathbb{R}^n} \frac{z_1}{|\mathbf{z}|^{n-(\alpha-2)}} \cdot \tilde{\rho}(\mathbf{z})\ d_\omega \mathbf{z},$$

$$(\mathcal{I}_\alpha \rho)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\tilde{\rho}(\mathbf{z})}{|\mathbf{z}|^{n-\alpha}} d_\omega \mathbf{z}$$

$$(\mathcal{I}_{\alpha-2} \rho)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\tilde{\rho}(\mathbf{z})}{|\mathbf{z}|^{n-\alpha+2}} d_\omega \mathbf{z}$$

In other words, if you have a 'moment-like' inequality for the L.H.S., it implies the inequality for the gradient $\nabla \mathcal{I}_\alpha$.

Gradient estimates

Notice that a simple estimate implies

$$\frac{1}{n-\alpha}\left|(\nabla \mathcal{I}_\alpha \rho)(\mathbf{x})\right| = \left|\int_{\mathbb{R}^n} \frac{z_1}{|\mathbf{z}|^{n+2-\alpha}} \cdot \tilde{\rho}(\mathbf{z})\, d_\omega \mathbf{z}\right| \le \int_{\mathbb{R}^n} \frac{1}{|\mathbf{z}|^{n-(\alpha-1)}} \cdot \tilde{\rho}(\mathbf{z})\, d_\omega \mathbf{z} = \mathcal{I}_{\alpha-1}\tilde{\rho} \tag{3}$$

and, similarly, the Cauchy inequality with $z_1 \le 1 \cdot |\mathbf{z}|$ implies that

$$\frac{1}{n-\alpha}|\nabla \mathcal{I}_\alpha \rho| \le \sqrt{\mathcal{I}_\alpha \tilde{\rho} \cdot \mathcal{I}_{\alpha-2}\tilde{\rho}}. \tag{4}$$

But these inequalities **are far from being optimal**. Indeed, the above estimate (1) gives:

$$\sinh^2\left(\frac{1}{2}\int_{\mathbb{R}+} \frac{\rho(x)}{x}dx\right) \le \frac{1}{4}\int_{\mathbb{R}+} \rho(x)dx \int_{\mathbb{R}+} \frac{\rho(x)}{x^2}dx,$$

which can be rewritten for $n = \alpha = 1$ (notice that $\omega_1 = \mu_1([-1,1]) = 2$) as

$$\int_{\mathbb{R}+} \frac{x\rho(x)}{x^2}dx_\omega \le \sinh^{-1}\left(\sqrt{\int_{\mathbb{R}+} \frac{\rho(x)}{x^0}dx_\omega \int_{\mathbb{R}+} \frac{\rho(x)}{x^2}dx_\omega}\right) = \sinh^{-1}\sqrt{\mathcal{I}_1\rho \cdot \mathcal{I}_{-1}\rho}.$$

Similarly, interpreting

$$\frac{1}{2}\left(\int_{\mathbb{R}} \rho(x)dx\right)^2 \leq \tanh\left(\frac{1}{2}\int_{\mathbb{R}} \frac{\rho(x)}{|x|}dx\right) \cdot \int_{\mathbb{R}} |x|\rho(x)dx.$$

for $n = 1$ and $\alpha = 2$, we obtain

$$\left(\int_{\mathbb{R}+} \frac{x\rho(x)}{x^1}dx_\omega\right)^2 \leq \tanh\left(\int_{\mathbb{R}+} \frac{\rho(x)}{x^1}dx_\omega\right) \cdot \int_{\mathbb{R}+} \frac{\rho(x)}{x^{-1}}dx_\omega,$$

implying an exact estimate

$$\boxed{|\nabla \mathcal{I}_2\rho|^2 \leq \tanh \mathcal{I}_0\rho \cdot \mathcal{I}_2\rho}$$

This suggests a different **shape** of the corresponding inequality, we discuss this below.

Gradient estimates for general dimensions

Given a measurable function $0 \le \rho(x) \le 1$, $x \in \mathbb{R}^n$, and $0 \notin \operatorname{supp}\rho$, find a **sharp** inequality which involves

$$u := \mathcal{I}_\alpha \rho, \quad v := \mathcal{I}_{\alpha-2}\rho, \quad \text{and} \quad w := \frac{1}{n-\alpha}|\nabla \mathcal{I}_\alpha \rho|.$$

In other words, we want to determine

$$\mathscr{N}_\alpha(u,v) := \sup_\rho \left\{ w^2 : \ \mathcal{I}_\alpha \rho = u, \ \ \mathcal{I}_{\alpha-2}\rho = v \right\}.$$

In this notation,

$$\frac{1}{n-\alpha}|\nabla \mathcal{I}_\alpha \rho| \le \sqrt{\mathscr{N}_\alpha\big(\mathcal{I}_\alpha\rho, \mathcal{I}_{\alpha-2}\rho\big)}.$$

A pair $(u,v) \subset \mathbb{R}^2_{\ge 0}$ is said to be *admissible* if $\exists \rho$: $0 \le \rho \le 1$:t $\mathcal{I}_\alpha \rho = u$ and $\mathcal{I}_{\alpha-2}\rho = v$.

Some natural questions arise:

- How does the **shape** of the goal function $\mathscr{N}_\alpha(u,v)$ depend on u and v?
- Does $\mathscr{N}_\alpha(u,v)$ separate into functions of u and v for a general α?
- When it is symmetric in u and v?

- V. Tkachev, Sharp pointwise gradient estimates for Riesz potentials with a bounded density, *Anal. Math. Physics, 8(2018)*

Theorem 2

Let $n \geq 1$ *and* $\alpha \in (0, 2]$*. Then the set of admissible pairs coincides with the nonnegative quadrant* $\mathbb{R}^2_{\geq 0}$ *and*

$$\boxed{\mathscr{N}_\alpha(u, v) = u^{2(\alpha-1)/\alpha} \frac{h_\alpha^2(t)}{f_\alpha^{2(\alpha-1)/\alpha}(t)}, \qquad \forall u, v > 0} \tag{5}$$

where $t = t(u, v)$ *is uniquely determined by the relation*

$$f_\alpha^{2-\alpha}(t) f_{\alpha-2}^{\alpha}(t) = u^{2-\alpha} v^\alpha, \tag{6}$$

where

$$f_\alpha(t) = t^{2-n}(t^2-1)^{n/2} F(\tfrac{2-\alpha}{2}, \tfrac{2+\alpha}{2}; \tfrac{n+2}{2}, 1-t^2)$$
$$h_\alpha(t) = t^{1-n}(t^2-1)^{n/2} F(\tfrac{2-\alpha}{2}, \tfrac{\alpha}{2}; \tfrac{n+2}{2}, 1-t^2),$$

and $F([a, b], [c], t)$ *is the Gauss hypergeometric function.*

The *shape structure* of the goal function $\mathscr{N}_\alpha(u,v)$ is still hidden, but one has

$$\mathscr{N}_\alpha(u,v) = u^{2(\alpha-1)/\alpha}\Phi_{n,\alpha}(u^{2-\alpha}v^{\alpha}).$$

Two particular cases are interesting for applications and can be simplified to

$$\text{for } \alpha = 2: \qquad \mathscr{N}_2(u,v) = u \cdot \phi_n(v),$$

$$\text{for } \alpha = 1: \qquad \mathscr{N}_1(u,v) = \psi_n(u \cdot v).$$

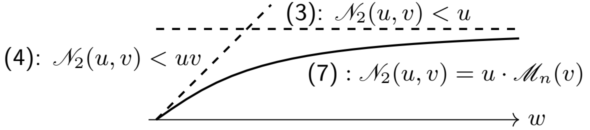
The case $\alpha = 2$ was the starting point for the above results. Let $\mathcal{M}_n(t)$ be the solution of $\mathcal{M}_n'(t) = 1 - \mathcal{M}_n^{2/n}(t)$, $\mathcal{M}(0) = 0$. For example, $\mathcal{M}_1(t) = \tanh t$, $\mathcal{M}_2(t) = 1 - e^{-t}$.

Theorem 3 ($\alpha = 1$, V.T., 2005)

For any density function $0 \le \rho(x) \le 1$, $0 \notin \operatorname{supp}\rho$

$$\left(\int_{\mathbb{R}^n} \frac{x_1\rho(x)}{|x|^n}\, d_\omega x\right)^2 \le \mathcal{M}_n\left(\int_{\mathbb{R}^n} \frac{\rho(x)}{|x|^n}\, d_\omega x\right)\int_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n-2}}\, d_\omega x. \tag{7}$$

The inequality is **sharp** *and the equality holds when* $\rho(x)$ *is the characteristic function of ball* B *with a center on the* x_1*-axes and* $0 \notin \overline{B}$.



Theorem 4 ($\alpha = 2$, V.T., 2018)

For any measurable function $0 \le \rho(x) \le 1$, $0 \notin \operatorname{supp}\rho$, *the sharp inequality holds*

$$\left| \int_{\mathbb{R}^n} \frac{x_1 \rho(x)}{|x|^{n+1}} \, d_\omega x \right| \le \Phi_n \left(\sqrt{\int_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n-1}} \, d_\omega x \cdot \int_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n+1}} \, d_\omega x} \right), \tag{8}$$

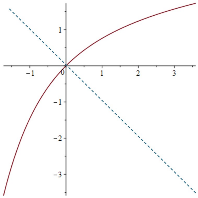
where $\Phi_n(s)$ *is the unique solution of the initial problem*

$$\Phi_n'' = \frac{\Phi_n'(\Phi_n'^2 - 1)}{(n-1)\Phi_n \Phi_n' + s}, \qquad \Phi_n(0) = 0, \;\; \Phi_n'(0) = 1 \tag{9}$$

subject to the asymptotic condition

$$\lim_{s \to \infty} \frac{\Phi_n(s)}{\ln s} = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{3}{2})}. \tag{10}$$

The case $\alpha = 1$ and $n = 2$



The case $n = 2$ is even very special among $\alpha = 2$. Both $\Phi_2(s)$ and its inverse satisfy the same ODE

$$\Phi_2'' = \frac{\Phi_2'(\Phi_2'^2 - 1)}{\Phi_2\Phi_2' + s}, \qquad \Phi_2(0) = 0,\ \ \Phi_2'(0) = 1$$

Furthermore, the function Φ_2 has some extra symmetries and a nice parameterizations by virtue of complete elliptic integrals:

$$[s(k), \Phi(k)] = [\frac{4}{\pi}(E(k) - K(k)),\ \frac{4}{\pi}(k \cdot K(k) - \tfrac{1}{k} \cdot E(k))],$$

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-(1-k^2)t^2}}, \qquad E(k) = \int_0^1 \frac{\sqrt{1-(1-k^2)t^2}}{\sqrt{1-t^2}}dt$$

The following remarkable properties: the Taylor expansion at the origin of $\Phi_2(z)$ is

$$\Phi_2(z) = z - \tfrac{1}{2^2}z^2 + \tfrac{1}{2^4}z^3 - \tfrac{7}{2^9}z^4 + \tfrac{5}{2^{11}}z^5 - \tfrac{21}{2^{16}}z^6 + \tfrac{3}{2^{17}}z^7 + \tfrac{7}{2^{24}}z^8 + \tfrac{11}{2^{26}}z^9 - \tfrac{959}{2^{32}}z^{10} + \dots$$

and $\Phi(z)$ satisfies the following **involutive property**:

$$\boxed{(-\Phi_2) \circ (-\Phi_2) = \mathrm{id}.}$$

We don't yet know any conceptional explanation of these facts.

The classical L-problem of moments

A truncated moment problem with bounds appears in A. A. Markov (1896) research on probability theory. Krein and Akhieser (in 1930's) considered the following problem: given a measurable function $0 \le \rho(x) \le L$ on $\mathbb{R}$ define its moments

$$\ell_n := \int_{\mathbb{R}} x^n \rho(x)\, dx.$$

How much it could be said about ρ if only finitely many moments are known? Which ρ are finitely determined?... The truncated moment problem can be formulated in $\mathbb{R}^n$.

Some applications/motivations

- Probability (reconstruction of probability density functions)
- Physics (determination of contours)
- Subnormal operator theory
- Computer Science (image recognition and reconstruction)
- Geography (location of proposed distribution centers)
- Environmental Science (oil spills, via quadrature domains)
- Engineering (tomography)
- Optimization (finding the global minimum of a real polynomial in several real variables)
- Function Theory (a dilation-type structure theorem in Fejér-Riesz factorization theory)
- Geophysics (inverse problems, cross sections)

Typical Problem: Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.

The classical L-problem of moments

The **exponential transform** of $\rho(x)$ is the formal series identity:

$$\exp\left(-\frac{1}{L}\sum_{n=1}^{\infty}\ell_{n-1}t^{-n}\right) = \exp\left(-\frac{1}{L}\int_{\mathbb{R}}\frac{\rho(\zeta)d\zeta}{t-\zeta}\right)$$
$$= 1 + \frac{\sigma_0}{t} + \frac{\sigma_1}{t^2} + \ldots := E(\rho(x), t)$$

Example. If $\rho(x) = \chi_{[a,b]}(x)$ then $\ell_{n-1} = \frac{b^n - a^n}{n}$, $n \geq 1$, therefore

$$\sum_{n=1}^{\infty}\ell_{n-1}t^{-n} = \ln\frac{t-a}{t-b}$$
$$E(\chi_{[a,b]}(x), t) = \frac{t-b}{t-a} = 1 + \frac{b-a}{t}\sum_{n=0}^{\infty}\frac{b^n}{t^n}$$

In general, for union of several intervals, $E(\chi_D(x), t) = \prod \frac{t-b_i}{t-a_i}$ is a **rational function**, in particular, the Hankel determinant sequence

$$\Delta_i := \det(\sigma_{i+j})_{0\leq i,j\leq N}$$

vanishes after some integer N. For example, for one interval,

$$\Delta_1 = S, \qquad \Delta_2 = \begin{vmatrix} S & Sb \\ Sb & Sb^2 \end{vmatrix} = 0, \quad \text{etc.,} \qquad S := b - a$$

Andrei Andreyevich Markov (1856-1922)

Naum Il'ich Akhiezer (1901-1980)

Mark Grigorievich Krein (1907-1989)

Adolf Abramovich Nudelman (1931-2011)

Theorem (Krein, Akhieser, Nudelman)

- The L-**problem** has a solution if and only if the sequence (σ_n) is **nonnegative definite** on $\mathbb{R}$ (i.e. $(\sigma_{i+j})_{0\leq i,j\leq N}$ is nonnegative definite).
- A class of **extremal** solutions of the L-problem (in the natural convex set of solutions) corresponds to **degenerated** non-negative definite sequences (σ_n), i.e. $\exists N$: $\det(\sigma_{i+j})_{0\leq i,j\leq N} = 0$.
- Any extremal solution is (proportional to) the characteristic function of a union of at most N disjoint bounded intervals: $\rho = L \cdot \chi_{\cup \Delta_i}$.

If $L = 1$ and $I = [0, \infty)$, the solvability of the corersponding L-moment problem is equivalent to that of the Stieltjes problem for $\{\sigma_k\}_{k\geq 0}$, which is equivalent to the nonnegativity of the Hankel criterium

$$\Delta_m := \det(\sigma_{i+j})_{i,j=0}^m \geq 0, \qquad \Delta'_m := \det(\sigma_{i+j+1})_{i,j=0}^m \geq 0, \quad m \geq 0.$$

Example. For the exponential transform this readily yields (Markov's inequalities for L-moments)

$$\ell_0^4 \leq 12(\ell_0\ell_2 - \ell_1^2), \quad \text{etc (higher terms inequalities)} \tag{11}$$

Comparing the above example when $\rho = \chi_{[a,b]}(x)$ and $\ell_{n-1} = (b^n - a^n)/n$, this becomes

$$(b-a)^4 \leq 12((b-a)\cdot \frac{b^3-a^3}{3} - \frac{(b^2-a^2)^2}{4})\ldots = (b-a)^4$$

The general exponential transform in $\mathbb{R}^n$

The exponential transform can be viewed as a potential depending on a domain in $\mathbb{R}^n$, or more generally on a measure having a *density* function $\rho(x)$ (with compact support) in the range $0 \leq \rho \leq 1$. More precisely,

$$E_\rho(x) = \exp\left[-\frac{2}{n\omega_n}\int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}\right],$$

If $\rho(x) = \chi_D(x)$ (the most interesting and, in a sense, an extremal case) then

$$E_D(x) = \exp\left[-\frac{2}{n\omega_n}\int_D \frac{d\zeta}{|x-\zeta|^n}\right].$$

The 2D exponential transform and quadrature domains

The 2D-version has appeared in the 1970s in **operator theory**, as a **principal** function of certain close to normal operators and has been intensively studied by many researchers

- J.D. Pincus, Commutators and systems of singular integral equations, Acta Math., 121 (1968).
- J.W. Helton, R.E. Howe, Traces of commutators of integral operators, Acta Math., 135 (3–4) (1975)

More precisely, for any measurable function

$$\rho : \mathbb{C} \to [0, 1]$$

of compact support there exists a unique irreducible, linear bounded operator T acting on a Hilbert space H, with **rank-one self-commutator** $[T^*, T] = \xi \otimes \xi$, which factors E_ρ as

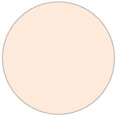
$$E_\rho(z, w) = \exp\left[-\frac{1}{\pi}\int \frac{\rho(\zeta)\, dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right] = 1 - \langle (T^* - \bar{w})^{-1}\xi,\, (T^* - \bar{z})^{-1}\xi\rangle \qquad (12)$$

The 2D-exponential transform has also recently been proved to be useful within **operator theory, moment problems** and other problems of **domain identification**, and for proving regularity of free boundaries (Gustafsson, Putinar, Milanfar, Shahgholian,. . .).

The 2D exponential transform and quadrature domains

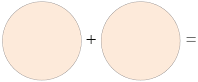
If one would infer from the 1D-picture a good class of **extremal** domains for Markov's L-problem in 2D, one would choose the disjoint **unions of disks**, as immediate analogs of disjoint **unions of intervals**. In reality, the nature of the complex plane is more complicated.

Example 1: the unit disk $\Omega = \mathbb{D}(0,1)$:



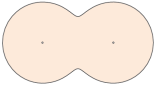
$$E_{\mathbb{D}} = 1 - \frac{1}{z\bar{w}}.$$

Example 2: $\Omega = \mathbb{D}(-1,1) \cup \mathbb{D}(1,1)$:



$$E_{\Omega} = \left(1 - \frac{1}{(z+1)(\bar{w}+1)}\right)\left(1 - \frac{1}{(z-1)(\bar{w}-1)}\right).$$

Example 3: $\Omega = \mathbb{D}(-1,r) \oplus \mathbb{D}(1,r)$, $r > 1$ (a quadrature domain, see below):



$$E_{\Omega} = 1 - \frac{1 + A(r)z\bar{w}}{(\bar{w}^2-1)(z^2-1)}.$$

The 2D exponential transform and quadrature domains

- D. Aharonov, H. S. Shapiro, Domains on which analytic functions satisfy quadrature identities, J. Anal. Math., 30 (1976)
- B. Gustafsson, Quadrature identities and the Schottky double, Acta Appl. Math.1 (1983)
- M. Putinar, Linear analysis of quadrature domains, Ark. Mat. 33 (1995).

Ω := a *quadrature domain* (for analytic functions) if

$$\iint_\Omega h\,dxdy = \sum_{i=1}^{n} c_i h(z_i) \quad \forall h \in L^1(\Omega), \quad \text{for fixed } z_i \in \Omega, c_i \in \mathbb{C}.$$

The **exponential transform** of a bounded closed set Ω is defined by

$$E_\Omega(z,w) = \exp\left(-\frac{1}{\pi}\iint_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right) = 1 - \sum_{m,n=0}^{\infty} \frac{\sigma_{m,n}}{z^{m+1}\bar{w}^{n+1}}.$$

It connects the exponential and the **complex moments** $\ell_{m,n} = \iint_\Omega \zeta^m \bar{\zeta}^n dA(z)$.

$$\sum_{m,n=0}^{\infty} \frac{\sigma_{m,n}}{z^{m+1}\bar{w}^{n+1}} = 1 - \exp(-\sum_{m,n=0}^{\infty} \ell_{m,n} z^{m+1}\bar{w}^{n+1}),$$

The 2D exponential transform and quadrature domains

Theorem. (Aharonov-Shapiro 1976; Gustafsson 1983, Putinar 1996)

The following conditions are equivalent:

- $E_\Omega(z,w)$ is rational $= \frac{Q(z,w)}{P(z)\overline{P(w)}}$, $|z|, |w| \gg 1$;
- Ω is a quadrature domain;
- Ω is determined by finitely many moments $\ell_{jk} \Leftrightarrow \det(\sigma_{jk})_0^N = 0$ for some N;
- There is a bounded linear operator T acting on a Hilbert space, with spectrum equal to Ω, with rank one self commutator $[T^*, T] = \langle \xi \otimes \xi \rangle$ and such that the linear span $(T^{*k}\xi)_{k\geq 0}$ is finite dimensional.

Theorem (B. Gustafsson, 1983)

A domain Ω is a quadrature domain if and only if its Schwarz function is meromorphic on the Schottky double of Ω. A boundary is **always algebraic**.

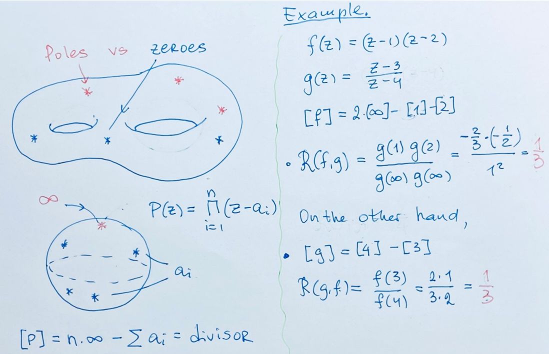
Theorem (M. Putinar, 1996)

There is a bounded linear operator T acting on a Hilbert space, with spectrum equal to Ω, with rank one self commutator $[T^*, T] = \xi \otimes \xi$ and such that the linear span $(T^{*k}\xi)_{k\geq 0}$ is **finite dimensional**. In particular, $E_\Omega(z,w)$ is **rational**.

Given meromorphic functions f, g on a compact Riemann surface M, their **meromorphic resultant** is

$$\mathcal{R}_M(f,g) = \mathcal{R}(f,g) = \frac{g(f^{-1}(0))}{g(f^{-1}(\infty))} = \prod_{i=1}^{m} \frac{g(a_i)}{g(\sigma_i)}$$

where $(f) = \sum a_i - \sum \sigma_i$ is the principal divisor of f. By Weil's reciprocity law, $\mathcal{R}(f,g) = \mathcal{R}(g,f)$.



Then *the exponential transform of a quadrature domain* Ω *is the meromorphic resultant* on the Schottky double $\hat{\Omega}$:

$$E_\Omega(z,w) = \mathcal{R}_{\hat{\Omega}}(f - z, g - \bar{w}).$$

- B. Gustafsson, V.Tkachev, The Resultant on Compact Riemann Surfaces, *Comm. Math. Phys., 2009*

Some very recent application

Arctic curves of periodic dimer models and generalized discriminants by Mateusz Piorkowski, arXiv:2410.17138, Submitted on 22 Oct 2024

Abstract: *We compute the algebraic equation for arctic curves of the Aztec diamond with a doubly (quasi-)periodic weight structure and obtain similar results for certain models of the hexagon [. . .] The key to our result is the construction of a discriminant for meromorphic differentials on a higher genus Riemann surface. This construction works analogously for meromorphic sections of arbitrary holomorphic line bundles. . .*

Dear Björn, dear Vladimir,

I have recently uploaded a paper on the arxiv which might be of interest to you. It deals mainly with the construction of discriminants on Riemann surfaces, but I also discuss resultants in Section 6. The main motivation to study these objects comes from statistical physics, more precisely dimer (tiling) models and the arctic curves phenomenon. [. . .] There might be some connection to your paper "The resultant on compact Riemann surfaces" though the actual definition and construction that I use differs substantially. Nonetheless, I thought you might be interested. [. . .]

Best/Mateusz,

2024/10/28

properties of the inverse Kasteleyn matrix. A similar approach can be formulated for models of the Aztec diamond and the hexagon using a bijection between tilings and families of nonintersecting paths. Here, the Eynard–Mehta Theorem [21] gives us the determinantal structure, see also [20, Sect. 4]. A comparison between the nonintersecting paths method and the inverse Kasteleyn matrix method can be found in [14, Sect. 4].

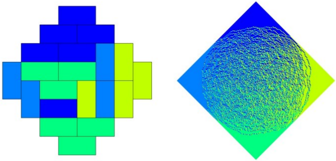


FIGURE 1. A random tiling of the Aztec diamond of size 4 (left) and of size 200 (right). Note the frozen regions in the corners and the emergence of a disc-shaped rough region in the middle. In this case the arctic curve becomes a circle as proven by Jockusch, Propp and Shor [30] (generated using code kindly provided by Christophe Charlier).

As shown in the seminal work of Kenyon, Okounkov and Sheffield [36] universal behavior for the height function fluctuations emerge under the assumption that the weights have a doubly periodic structure. In particular, the local dimer statistics converge to a translation invariant Gibbs ensemble of which there are three types:

- Gustafsson, B. and Putinar, M., The exponential transform: a renormalized Riesz potential at critical exponent, *Indiana Univ. Math. J. 52 (2003)*

Gustafsson and Putinar considered the n-dimensional version

$$E_\rho(x) = \exp\left[-\frac{2}{n}\int_{\mathbb{R}^n} \frac{\rho(\zeta)d_\omega\zeta}{|x-\zeta|^n}\right] = \exp(-\frac{2}{n}\mathcal{I}_0(x)), \qquad d_\omega\zeta := \frac{1}{\omega_n}d\zeta.$$

and proved that although the Riesz potential produces a **logarithmic singularity** at x when this variable tends from outside to a smooth portion of the boundary $\partial\,\mathrm{supp}\,\rho$, the exponential restores the smoothness in x, even up to **real analyticity**.

In the same paper, they proved that for $n = 2$, $\ln(1 - E_\Omega(x))$ is a **subharmonic function** for all $x \notin \Omega$. The proof makes use some integral representations and Ahlfors-Beurling capacity estimates. They also deduce an asymptotic decomposition for $\Omega \subset \mathbb{R}^n$:

$$1 - E_\Omega(x) = \frac{2|\Omega|}{|S^{n-1}|\cdot|x|^n} + O(\frac{1}{|x|^{n+1}})$$

and conjectured that a much stronger statement should holds:

Conjecture. In the above notation,

$$\begin{cases} \ln(1-E_\rho), & \text{if} \quad n=2, \\ \frac{1}{n-2}(1-E_\rho)^{(n-2)/n}, & \text{if} \quad n \geq 3, \end{cases}$$

is subharmonic outside $\operatorname{supp}\rho$ for any density $\rho \not\equiv 0$.

The above conjecture essentially claims that the function

$$\Phi_n^{(n-2)/n}(\mathcal{I}_0\rho(x)) := (1-\exp(-\frac{2}{n}\mathcal{I}_0\rho(x)))^{(n-2)/n}$$

is subharmonic for $n \geq 3$, where

$$\Phi_n(t) = 1 - e^{-2t/n}.$$

A refinement of $(1-E_\rho)^{(n-2)/n}$ is an arbitrary function $F(\mathcal{I}_0\rho)^{(n-2)/n}$ satisfying $0 \leq F(t) < 1$ for any $t \geq 0$. Then

The subharmonicity

- V. Tkachev, Subharmonicity of higher dimensional exponential transforms, Operator Theory: Advances and App., (156)2005.

$$\begin{aligned}
\Delta F(\mathcal{I}_0)^{\frac{n-2}{n}} &= \frac{n-2}{n}(F'(\mathcal{I}_0)F(\mathcal{I}_0)^{-\frac{2}{n}}\Delta\mathcal{I}_0 + (F'(\mathcal{I}_0)F(\mathcal{I}_0)^{-\frac{2}{n}})'|\nabla\mathcal{I}_0|^2) \\
&\ldots(\text{now choose } F \text{ such that } F' = 1 - F^{2/n})\ldots \\
&= \frac{n-2}{n^2}(1 - F(\mathcal{I}_0)^{\frac{2}{n}})\left(nF(\mathcal{I}_0)^{-\frac{2}{n}}\Delta\mathcal{I}_0 - 2F(\mathcal{I}_0)^{-\frac{2+n}{n}}|\nabla\mathcal{I}_0|^2\right) \\
&= 2(n-2)(1 - F(\mathcal{I}_0)^{\frac{2}{n}})\left[F(\mathcal{I}_0)B - |A|^2\right],
\end{aligned} \tag{13}$$

where

$$A = \int \frac{(x-\zeta)\rho(\zeta)}{|x-\zeta|^{n+2}}\, d\zeta_\omega,$$

$$B = \int \frac{\rho(\zeta)}{|x-\zeta|^{n+2}}\, d\zeta_\omega,$$

$$\mathcal{I}_0 = \int \frac{\rho(\zeta)}{|x-\zeta|^{n}}\, d\zeta_\omega.$$

Therefore the sign of the Laplacian $\Delta F(x)$ coincides with the sign of $F(\mathcal{I}_0)B - |A|^2$.

The subharmonicity

Theorem (V.T., 2005) If $0 \leq \rho(x) \leq 1$ with **compact** support, $0 \notin \operatorname{supp} \rho$, $x \in \mathbb{R}^n$, $\mathcal{M}_n'(t) = 1 - \mathcal{M}_n^{2/n}$ then

$$\mathcal{M}_n(\mathcal{I}_0)B - |A|^2 \geq 0.$$

One of the key elements in the proof was the following relation

$$\mathcal{M}_n\left(\int_{B(R)} \frac{d_\omega \zeta}{|x-\zeta|^n}\right) = \frac{R^n}{|x|^n}, \tag{14}$$

in other words, $\mathcal{M}_n$ transfers the $\alpha = 0$ index Riesz potential of any ball to **its kernel** (up to a normalization).

- **Remark 1.** The subharmonicity of $\frac{1}{n-2}(1-E_\rho)^{(n-2)/n}$ is weaker than that of $\mathcal{M}_n(E_\rho)$, and it readily follows from

$$\mathcal{M}_n(w) \le \frac{e^{2w/n}-1}{e^{2w/n}-\frac{n-2}{n}}.$$

- **Remark 2.** Using the inversion $x \to x/|x|^2$, the desired inequality follows from a particular one:

$$\left(\int \frac{x_1\rho(x)}{|x|^n}\,d_\omega x\right)^2 \le \mathcal{M}_n\left(\int \frac{\rho(x)}{|x|^n}\,d_\omega x\right)\int \frac{\rho(x)}{|x|^{n-2}}\,d_\omega x \tag{15}$$

- **Remark 3.** Using an elementary inequality

$$\frac{(a-b)^2}{c+d} \le \max\left[\frac{a^2}{c},\frac{b^2}{d}\right], \qquad a,b,c,d>0,$$

it suffices to verify that (15) oven holds when $0 \le \rho(x) \le 1$ with compact support in the half-space: $\operatorname{supp}\rho \subset \mathbb{R}^n_+$.

Proposition. Let $0 \le \rho(x) \le 1$ have a compact support $\operatorname{supp}\rho \subset \mathbb{R}^n_+$ and $\mathcal{M}'_n(t) = 1-\mathcal{M}_n^{2/n}$. Then

$$\mathcal{N}_2(u,v) := \sup\left\{\left(\int \frac{x_1\rho(x)}{|x|^n}\,dx_\omega\right)^2 : \rho \in K(u,v)\right\} = u\mathcal{M}_n(v)$$

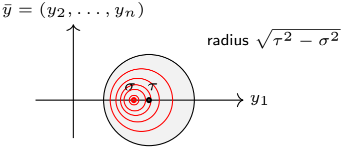
where

$$K(u,v) := \{\rho:\ 0 \le \rho(x) \le 1,\ \operatorname{supp}\rho \subset \mathbb{R}^n_+,\ \int \frac{\rho(x)}{|x|^{n-2}}\,dx_\omega = u,\ \int \frac{\rho(x)}{|x|^n}\,dx_\omega = v\}$$

Riesz potentials for a ball: a trick with co-area formula

Let us consider a ball

$$\begin{aligned} B(\tau,\sigma) &= \{y \in \mathbb{R}^n : |y - \tau e_1|^2 \le \tau^2 - \sigma^2\} \\ &= \{y \in \mathbb{R}^n : |y|^2 - 2\tau y_1 + \sigma^2 \le 0\} \\ &= \{y \in \mathbb{R}^n : \frac{\sigma}{\tau} \le \lambda(y) \le 1\} \end{aligned}$$



where the function $\lambda(y) = \frac{|y|^2+\sigma^2}{2\tau y_1}$ foliates $B(\tau,\sigma)$ into spheres

$$\partial B(\tau z,\sigma) : \{y \in \mathbb{R}^n : \ |y|^2 - 2\tau z y_1 + \sigma^2 = 0\} = \{y \in \mathbb{R}^n : \ |y - \tau z e_1|^2 = \tau^2 z^2 - \sigma^2\}$$

with moving centres at $(\tau z, 0)$ for $\frac{\sigma}{\tau} \le z < 1$. By the **co-area formula**

$$I := \int_{B(\tau,\sigma)} \frac{d_\omega y}{|y|^n} = \frac{1}{\omega_n} \int_{\sigma/\tau}^{1} dz \int_{\partial B(\tau z,\sigma)} \frac{dS}{|y|^n |\nabla \lambda(y)|}, \tag{16}$$

where

$$|\nabla \lambda|^2 = \frac{|\bar{y}|^2}{\tau^2 y_1^2} + \frac{(y_1^2 - \sigma^2 - |\bar{y}|^2)^2}{4\tau^2 y_1^4} \quad \Rightarrow \quad |\nabla \lambda|_{\partial B(\tau z,\sigma)} = \frac{\sqrt{\tau^2 z^2 - \sigma^2}}{\tau} \cdot \frac{1}{y_1}.$$

Substitution into (16) yields by the **harmonicity** of $y_1 |y|^{-n}$ and the **mean value property**

$$I = \int_{\sigma/\tau}^{1} \frac{\tau\, dz}{\sqrt{\tau^2 z^2 - \sigma^2}} \int_{\partial B(\tau z,\sigma)} \frac{y_1}{\omega_n |y|^n} dS = \int_{\sigma/\tau}^{1} \frac{\tau\, dz}{\sqrt{\tau^2 z^2 - \sigma^2}} \Big(\frac{\tau z}{(\tau z)^n} \cdot \frac{n\omega_n (\sqrt{\tau^2 z^2 - \sigma^2})^{n-1}}{\omega_n}\Big)$$

$$(\text{substitution } \tau z = \sigma \cosh t) = n \int_0^{\xi} \tanh^{n-1} t\, dt =: T_n(\xi), \qquad \text{where } \cosh \xi = \tau/\sigma.$$

An equivalent way to rewrite the latter result is

$$\mathcal{M}_n\left(\int_{B(R)} \frac{d_\omega \zeta}{|x-\zeta|^n}\right) = \left(\frac{R}{|x|}\right)^n, \tag{17}$$

in other words, $\mathcal{M}_n$ transfers the Riesz potential of index 0 to **its kernel** (up to a normalization).

Note that if $g(t) = (\tanh t)^n$ then $T_n'(t) = n(\tanh t)^{n-1}$, $g'(t) = n(\tanh t)^{n-1}(1-\tanh^2 t)$, hence $\frac{dg}{dT_n} = 1 - g^{2/n}$, which readily implies

$$g(t) = (\tanh t)^n = \mathcal{M}_n(T_n(t)). \tag{18}$$

In summary, using $\cosh \xi = \tau/\sigma$ and $\tau^2 - \sigma^2 = \sigma^2 \sinh^2 \xi$, we obtain using the Mean Value Property:

$$\int_{B(\tau,\sigma)} \frac{1}{|x|^n}\, dx_\omega = T_n(\xi) = v, \tag{19}$$

$$\int_{B(\tau,\sigma)} \frac{1}{|x|^{n-2}}\, dx_\omega = (\text{MVP}) = \sigma^2 \frac{\sinh^n \xi}{\cosh^{n-2} \xi} = \tau^2 (\tanh \xi)^n = \tau^2 \mathcal{M}_n(T_n(\xi)) = \tau^2 \mathcal{M}_n(v) = u, \tag{20}$$

$$\int_{B(\tau,\sigma)} \frac{x_1}{|x|^n}\, dx_\omega = (\text{MVP}) = \sigma \frac{\sinh^n \xi}{\cosh^{n-1} \xi} = \ldots = \tau \mathcal{M}_n(T_n(\xi)) = \tau \mathcal{M}_n(v) = \sqrt{u \mathcal{M}_n(v)}. \tag{21}$$

Note that the right hand side in (19) etc is increasing in ξ for any fixed σ, hence it takes any positive real value. By the above, $\widetilde{\rho}(x) := \chi_{B(\tau,\sigma)}(x) \in K(u,v)$, therefore

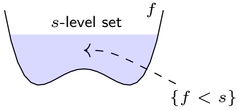
$$\mathcal{N}_2(u,v) \geq u \mathcal{M}_n(v).$$

'Bathtub principle'

Let f be μ-measurable, $\mu(\{x : f(x) = t\}) = 0$ and $\mu(\{x : f(x) < t\})$ be **finite** for all real t. A solution of the minimization problem

$$\inf\{\int f\rho\, d\mu : \rho \in \Omega(A)\},$$

where $\Omega(A) := \{\rho : 0 \le \rho \le 1, \int \rho\, d\mu = A\}$ is given by the characteristic function of a sublevel set $\rho_0 = \chi_{\{f<s\}}$ where $s = \sup\{t : \mu(\{x : f(x) < t\}) \le A\}$.



- **Remark 1.** An equivalent statement: in the above assumption,

$$\int_{\{t:f(t)<a\}} d\mu = \int \rho(x)\, d\mu \quad \text{implies} \quad \int_{\{t:f(t)<a\}} f(x)\, d\mu \le \int f(x)\rho(x)\, d\mu.$$

- **Remark 2.** Compare this with Markov-Chebyshev inequality $a \cdot \Pr(\xi > a) \le \mathrm{E}(\xi)$
- **Remark 3.** When $\mu(\{x : f(x) > t\})$ is finite for all real t, the corresponding statement holds for $\sup$.

Proof. Note that

$$f \cdot (\chi_{\{f<a\}} - \rho) = \begin{cases} f \cdot (0-\rho), & \text{if } f > a, \\ f \cdot (1-\rho), & \text{if } f < a. \end{cases} \le \begin{cases} a \cdot (0-\rho), & \text{if } f > a, \\ a \cdot (1-\rho), & \text{if } f < a. \end{cases} = a \cdot (\chi_{\{f<a\}} - \rho),$$

therefore

$$\int f \cdot (\chi_{\{f<a\}} - \rho)\, d\mu \le \int a \cdot (\chi_{\{f<a\}} - \rho) = 0$$

$\square$

Proof cont.: Conversely, given two arbitrary positive u, v, let τ, σ be chosen as in (19)–(21), and let $\widetilde{\rho}(x) := \chi_{B(\tau,\sigma)}(x)$. Consider $a = |x|^{2-n}$, $b = |x|^{-n}$, $c = x_1|x|^{-n}$. Define a measure on $\mathbb{R}^n$ by

$$d\mu = (a + \sigma^2 b)\, dx_\omega = (|x|^2 + \sigma^2)\frac{dx_\omega}{|x|^n}$$

Then for any test function $\rho \in K(u, v)$

$$\int \widetilde{\rho}\, d\mu = \int \rho\, d\mu = \int_{B(\tau,\sigma)} d\mu = \int (|x|^{2-n} + \sigma^2|x|^{-n})\, dx_\omega = u + \sigma^2 v \tag{22}$$

This means that ρ and $\widetilde{\rho}$ are also test functions for the extremal problem

$$\sup_\rho\{\int f\rho\, d\mu : \int \rho\, d\mu = u + \sigma^2 v\},$$

where $f(x) = \frac{c}{a+\sigma^2 b} = \frac{x_1}{|x|^2+\sigma^2} = \frac{1}{2\tau\lambda(x)}$. Then (a **magic point**!):

- $\{f(x) > \frac{1}{2\tau}\} = \{\lambda(x) < 1\} = B(\tau, \sigma)$,
- $\mu(\{f(x) > \frac{1}{2\tau}\}) = \mu(B(\tau, \sigma)) =$ (by (22)) $= u + \sigma^2 v$

By the Bathtub Principle, $\widetilde{\rho}$ is an extremal density, hence

$$\int f\rho\, d\mu \le \int f\widetilde{\rho}\, d\mu = \int_{B(\tau,\sigma)} f\, d\mu = \int_{B(\tau,\sigma)} x_1|x|^{-n}\, d\mu = \text{(by (21))} = \sqrt{u\mathcal{M}_n(v)}$$

therefore $\mathscr{N}_2(u, v) \le u\mathcal{M}_n(v)$. $\square$

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Sharp pointwise gradient estimates for Riesz potentials with a bounded density

Vladimir G. Tkachev[1]

Received: 26 March 2018 / Revised: 16 April 2018 / Accepted: 20 April 2018 /
Published online: 28 April 2018

Abstract We establish sharp inequalities for the Riesz potential and its gradient in $\mathbb{R}^n$ and indicate their usefulness for potential analysis, moment theory and other applications.

Keywords Riesz potentials · Exponential transform · L-problem of moments · Subharmonic functions · Cauchy's inequality · Symmetry of domains and solutions

Mathematics Subject Classification Primary: 31B15 · 47B06 · 44A60; Secondary: 44A15

1 Introduction

Given a measurable function $f(x)$ on $\mathbb{R}^n$, its Riesz potential of order $0 \le \alpha < n$ is defined[1] by

$$(\mathcal{I}_\alpha \rho)(y) = \int_{\mathbb{R}^n} \frac{\rho(x)}{|y-x|^{n-\alpha}}\, d_\omega x,$$

where $d_\omega x$ denotes the n-dimensional Lebesgue measure on $\mathbb{R}^n$ normalized by

$$d_\omega x = \frac{1}{\omega_n}\, dx,$$

[1] The normalization we use is slightly different from the standard normalization [3].

In memory of Sasha Vasil'ev Friend, Colleague, Mathematician.

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